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from a third particle are attracted by it with forces proportional to their masses" might be held to be merely a definition of mass. It might be asserted that "the statement that A's mass is three times as great as B's means merely that if A and B are equally distant from C the attractive force of C upon A is three times that of C upon B."

It is not difficult to show, however, that the conception of mass as a measure of quantity of matter is an essential part of the interpretation we put upon the law of gravitation. Thus if, when A and B are equally distant from C, the attraction of C upon A is three times that of C upon B, we say that it is because A has three times as much matter as B; and we infer that A might be divided into three parts each having a mass equal to that of B.

The law of gravitation thus derives an essential part of its significance from a notion of mass which is independent of that law—the notion that mass is a measure of the matter of which bodies are composed and satisfies the fundamental quantitative relations expressed by (I), (II) and (III). At the same time it must be recognized that the law itself gives precision to the definition and furnishes a method of comparing quantities of matter with exactness.

18. *Agreement of Two Methods of Estimating Quantity of Matter.*—It has been seen that the law of acceleration and the law of gravitation furnish two independent methods of applying exact quantitative measurements to matter. That the results of the two methods are in apparently exact agreement is a matter of experimental knowledge rather than of *a priori* necessity. If the fact were otherwise, weighing would not be an available method of comparing the masses of bodies for the purposes of dynamics.

GEOMETRICAL AND OTHER ILLUSTRATIONS OF INDETERMINATE FORMS.

By W. V. LOVITT, Purdue University.

In the AMERICAN MATHEMATICAL MONTHLY, Vol. III, 1893, pp. 207–8, there is given by B. F. Finkel a geometrical proof that $0 \times \infty$ is indeterminate. In the present paper additional illustrations, geometrical and otherwise, are given of this and other indeterminate forms.

I. The area generated by a segment of a straight line AB (Fig. 1) revolving about an axis CD in its plane is given by the formula:

$$\text{Area } AB = 2\pi \cdot AB \cdot MO = 2\pi \cdot CD \cdot MR,$$

where CD is the projection of the segment AB upon the line CD, M is the middle point of AB, O is the center of the segment CD, and MR is perpendicular to AB. Let AB revolve about M and become perpendicular to CD. Then MR becomes parallel to CD and $CD \cdot MR$ becomes $0 \times \infty$.

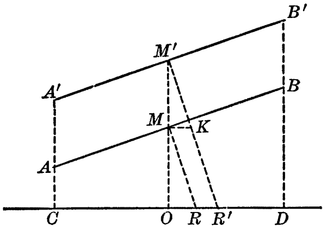

FIG. 1.

But $$CD \cdot MR = AB \cdot MO = K_1 \text{ (constant).}$$

Let us take a line segment $A'B' = AB$ and parallel to AB. Let M' be the middle point of $A'B'$, $M'R'$ perpendicular to $A'B'$ and parallel to MR. Then

$$\text{Area } A'B' = 2\pi \cdot CD \cdot M'R'.$$

Rotate $A'B'$ about M' until it is perpendicular to CD. Then $CD \cdot M'R'$ becomes $0 \times \infty$. But

$$CD \cdot M'R' = A'B' \cdot M'O = K_2 \text{ (constant).}$$

We note that $K_1 \neq K_2$; that is, our form $0 \times \infty$ takes on different values in these two instances and can thus be made to vary at will.

Draw MK parallel to CD. Consider the difference

$$\text{Area } A'B' - \text{Area } AB = 2\pi CD \cdot (M'R' - MR) = 2\pi CD \cdot M'K.$$

Let $A'B'$ and AB become perpendicular to CD. Then $M'R'$ and MR become parallel to CD and

$$M'R' - MR \text{ becomes } \infty - \infty.$$

But

$$M'R' - MR = M'K$$

and $M'K$ becomes infinite as $A'B'$ becomes perpendicular to CD. Hence, in this case the indeterminate form $\infty - \infty$ becomes infinite.

Incidentally, we have in the product $CD \cdot M'K$ an indeterminate form of the type $0 \times \infty$.

An additional illustration of the form $\infty - \infty$ is furnished by the hyperbola. This curve is defined as the locus of a point P, the difference of whose distances from two fixed points, F' and F, is a constant, $2a$. That is

$$F'P - FP = 2a.$$

As $F'P$ increases indefinitely, FP also increases indefinitely and

$$F'P - FP \text{ takes the form } \infty - \infty.$$

II. Consider now (Fig. 2) a triangle with a fixed base a, with sides x and y as indicated, the vertex P being on a line l which is parallel to the base. The area of this triangle is constant and will be denoted by c. Then

$$c = \frac{a}{2} x \sin \theta, \qquad \text{and} \qquad \frac{2c}{a} = x \sin \theta = x/\csc \theta.$$

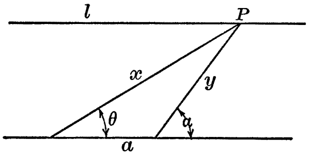

FIG. 2.

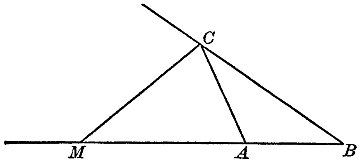

FIG. 3.

As P runs off to infinity on the line l, $x \sin \theta$ becomes $\infty \times 0$, while $x/\csc \theta$ becomes ∞/∞. From the law of sines,

$$\frac{x}{y} = \frac{\sin \alpha}{\sin \theta}.$$

As P moves off to infinity on the line l, the left and right hand members of the last equation take on the forms ∞/∞ and $0/0$ respectively. Thus we have two illustrations of the change from one indeterminate form to another.

III. The following theorem from plane geometry furnishes examples of the indeterminate forms ∞/∞ and $\infty - \infty$.

The bisector CM (Fig. 3) of an exterior angle of a triangle ABC divides the opposite side externally into segments which are proportional to the adjacent sides. That is,

$$\frac{MA}{MB} = \frac{CA}{CB}.$$

Let BC and AB remain fixed in length, and let AC increase, approaching BC in length. Then CM becomes parallel to AB, while MA/MB becomes ∞/∞. But

$$\lim_{CA \to BC} \frac{MA}{MB} = \lim_{CA \to BC} \frac{CA}{BC} = 1.$$

The difference $MB - MA$, which is always equal to the constant AB, becomes $\infty - \infty$.

IV. An additional illustration of the form $0 \times \infty$ is furnished by inversion with respect to a given circle, $x^2 + y^2 = a^2$. Let P and P' be inverse points and O the center of the given circle, then

$$\overline{OP} \cdot \overline{OP'} = a^2.$$

As OP approaches zero, OP' becomes indefinitely large and $\overline{OP} \cdot \overline{OP'}$ becomes $0 \times \infty$. In particular this form occurs in finding the inverse of a circle through the origin. The inverse is a straight line.

V. We add here two problems taken from works on hydraulics.

The theoretical efflux Q (Fig. 4) through a triangular orifice in a thin vertical plate or wall, with base, b, horizontal, is given by the following formula:

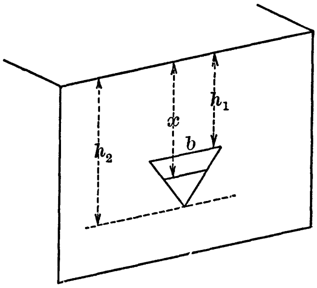

FIG. 4.

$$Q = \frac{2b\sqrt{2g}}{15}\,\frac{[2h_2^{5/2} - 5h_2h_1^{3/2} + 3h_1^{5/2}]}{(h_2 - h_1)}.$$

Let $h_1 = h_2$; then certainly $Q = 0$, while the expression for Q in the formula takes on the indeterminate form $0/0$.

Again, in hydrostatics the head h' on the center of pressure of a submerged rectangular plane, with edges parallel to the surface, one end having a head h_2, the other a head h_1, is

$$h' = \frac{2}{3}\cdot\frac{h_2^3 - h_1^3}{h_2^2 - h_1^2}.$$

When $h_2 = h_1$, $h' = h$, while the expression for h' in the formula becomes $0/0$.

VI. We conclude with a few simple arithmetical illustrations.

(1) Type $\frac{0}{0}$:	$\frac{1}{9}$; $\frac{.1}{.9}$; $\frac{.01}{.09}$; $\cdots$.	Constantly equal to $\frac{1}{9}$.
(2) Type $\frac{\infty}{\infty}$:	$\frac{3}{10}$; $\frac{33}{100}$; $\frac{333}{1000}$; $\cdots$.	Approaches $\frac{1}{3}$.

Any repeating decimal furnishes an additional example.

(3) Type $\infty - \infty$:	$1.3 - 1$; $2.33 - 2$; $3.333 - 3$; $\cdots$.	Approaches $\frac{1}{3}$.
(4) Type $0 \times \infty$:	1.2; $\frac{1}{2}\cdot 4$; $\frac{1}{4}\cdot 8$; $\cdots$.	Constantly equal to 2.
(5) Type ∞^0:	$(2^2)^{1/2}$; $(2^3)^{1/3}$; $(2^4)^{1/4}$; $\cdots$.	Constantly equal to 2.
(6) Type 0^0:	$(2^{-2})^{1/2}$; $(2^{-3})^{1/3}$; $(2^{-4})^{1/4}$; $\cdots$.	Constantly equal to $\frac{1}{2}$.